Probability

An introduction, mutually exclusive and independent events

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What is Probability? Probability measures **how likely** something is to happen, from:

- **0** (never happens)
- to **1** (always happens) (or 0% to 100%)

Rolling a fair die:

- $P(\text{Even}) = \frac{3}{6} = 0.5$
- $P(\text{Prime}) = \frac{3}{6} = 0.5$

Basic Probability Formula

$$P(\text{Event}) = \frac{\text{Number of ways it can happen}}{\text{Total possible outcomes}}$$

Important rules

Certain Event (100%)

$$P(\text{Sun rises}) = 1$$

Impossible Event (0%)

$$P(\text{Rolling a 7 on standard die}) = 0$$

Complement Rule ("Not")

$$P(\text{Not A}) = 1 - P(A)$$

Conditional Probability ("If...Then")

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Rules of probability

Rule	Formula
Complement	$P(A^c) = 1 - P(A)$
Union	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Mutually Exclusive	$P(A \cap B) = 0$
Independent	$P(A \cap B) = P(A)P(B)$

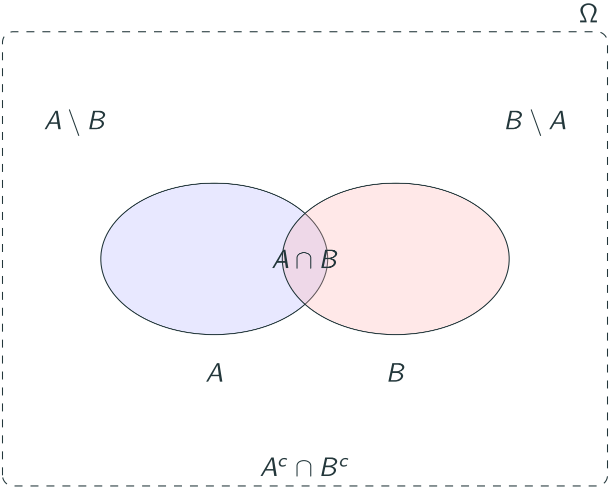
Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' Theorem originates from Conditional Probability

The probability of A *given* B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$


Ω
$A \setminus B$
$B \setminus A$
$A \cap B$
A
B
$A^c \cap B^c$

Real-world uses

- Medical diagnosis
- Spam filtering
- Risk assessment
- Machine learning (Naive Bayes)

Key Insight

- Conditional probability formalizes how we should rationally update beliefs given evidence.

Example 1

A family has two children. You know that at least one of them is a boy. What is the probability that both children are boys?

Example 1 - solution

Step 1: Sample Space The possible gender combinations for two children (older child first) are:

$$S = \{\mathrm{BB}, \mathrm{BG}, \mathrm{GB}, \mathrm{GG}\}$$

Step 2: Apply Condition We know *at least one* is a boy, so we eliminate GG:

$$S' = \{\mathrm{BB}, \mathrm{BG}, \mathrm{GB}\}$$

Step 3: Count Favorable Outcomes Only BB has two boys:

$$\text{Favorable} = 1 \quad \text{(BB)}$$

$$\text{Possible} = 3 \quad \text{(BB, BG, GB)}$$

Step 4: Calculate Probability

$$P(\text{Both boys}) = \frac{\text{Favorable}}{\text{Possible}} = \frac{1}{3}$$

Example 2

A bag contains 4 red marbles, 5 blue marbles, and 6 green marbles. If 3 marbles are randomly selected without replacement, what is the probability that all three are blue?

- $\frac{1}{91}$
- $\frac{2}{91}$
- $\frac{5}{182}$
- $\frac{10}{273}$
- $\frac{25}{546}$

Example 2 - solution Step 1: Determine Total Marbles First, find the total number of marbles:

$$4 \text{ red} + 5 \text{ blue} + 6 \text{ green} = 15 \text{ marbles}$$

Step 2: Calculate Total Possible Outcomes Number of ways to choose any 3 marbles from 15:

$$\text{Total combinations} = \binom{15}{3} = \frac{15!}{3!(15-3)!} = 455$$

Step 3: Calculate Favorable Outcomes Number of ways to choose 3 blue marbles from 5 available:

$$\text{Favorable combinations} = \binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$

Step 4: Compute Probability

$$P(3 \text{ blue}) = \frac{\text{Favorable}}{\text{Total}} = \frac{10}{455} = \frac{2}{91}$$

– (Final Answer)

Mutually Exclusive Events

Definition: Two events are mutually exclusive (or disjoint) if they **cannot happen at the same time**. If one occurs, the other cannot.

Examples:

- Flipping a coin: **Heads** and **Tails** are mutually exclusive (you can't get both at once).
- Rolling a die: Getting a **1** and a **6** are mutually exclusive (only one face lands up).

Properties of Mutually Exclusive events:

- If two events A and B are mutually exclusive:

$$P(A \cap B) = 0$$

- Their combined probability is:

$$P(A \cup B) = P(A) + P(B)$$

Warning: Mutually exclusive events are not independent (unless one has zero probability). Why? Because if A happens, B **must not** happen, meaning A affects B's probability.

Independent Events

Definition: Two events are independent if the occurrence of one **does not affect** the probability of the other. They have no influence on each other.

Examples:

- Flipping a coin twice: The first **Heads** doesn't change the chance of the second **Heads** (still 50%).
- Rolling a die and flipping a coin: The die outcome doesn't affect the coin's result.

Properties of Independent events:

- If A and B are independent:

$$P(A \cap B) = P(A) \times P(B)$$

- The probability of A given B is just $P(A)$:

$$P(A \mid B) = P(A)$$

Warning: Independent events are **not mutually exclusive** (unless one has zero probability). Why? Because if A and B can both happen, $P(A \cap B) \neq 0$.

Differences between mutually exclusive and independent events:

Mutually Exclusive	**Independent**
"Cannot happen together"	"Do not affect each other"
$P(A \cap B) = 0$	$P(A \cap B) = P(A)P(B)$
If A happens, B **cannot**.	If A happens, B's chance **stays the same**.

Final Intuition

- Mutually Exclusive: "Either this **or** that, but not both."
- Independent: "This happening doesn't change the odds of that happening."

Mutually Exclusive Events Scenario: Let A be the event "rolling a 3" on a die and B be the event "rolling a 4" on the same die.

- If A occurs (die shows 3), B **cannot** occur (die cannot simultaneously show 4).
- Thus, $P(A \cap B) = 0$.

Conclusion: A and B are **mutually exclusive** (disjoint).

Independent Events Scenario: Let C be the event "rolling a 5" on the die, and D be the event "flipping Heads" on the coin.

- The outcome of the die does **not** affect the coin flip, and vice versa.
- $P(C) = \frac{1}{6}$, $P(D) = \frac{1}{2}$.
- $P(C \cap D) = P(C) \times P(D) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

Conclusion: C and D are **independent**.

Next class we will cover tree and Venn diagrams.