Continuous Distributions

Uniform Distribution and Exponential Distribution

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What is a Probability Density Function

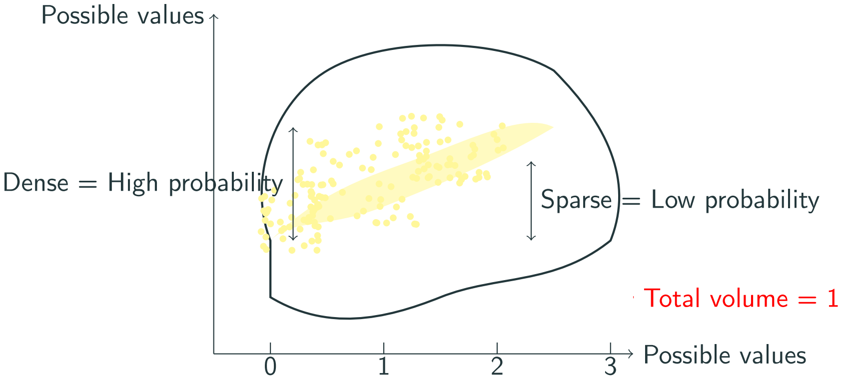
For continuous random variables (like height, temperature, or time), we use a **Probability Density Function (PDF)** because:

- We can't list probabilities for every value (infinitely many!)
- Instead, we describe *how densely packed* probabilities are

What is a Probability Density Function

- The **shape** shows where values are more/less likely
- The **total volume** is always 1 (100% probability)
- **Probabilities** are measured by *how much density* is in regions

What is a Probability Density Function



Probability Density Function

The Probability Density Function (PDF) of a continuous random variable X is a function $f_X(x)$ that describes the relative likelihood of X taking a given value x. The probability that X lies in an interval $[a, b]$ is given by:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

Uniform Distribution

A statistical experiment is defined as a uniform experiment when a continuous random variable X has a uniform distribution where $a \leq x \leq b$, then X takes on values between a and b. All values x are equally likely.

Uniform Distribution

- Distribution: $X \sim Uniform(a, b)$
  - a = the lower limit
  - b = the upper limit
- Mean: $\mu = \frac{a+b}{2}$
- Standard Deviation: $\sigma = \sqrt{\frac{(b-a)^2}{12}}$
- Probability Density Function: $f_x(x) = \frac{1}{b-a}$
- Probability: $P(X \leq x) = \frac{x-a}{b-a}$

Uniform Distribution

The probability $P(c \leq X \leq d)$ may be found by computing the area under $f(x)$, between c and d. Since the corresponding area is a rectangle, the area may be found simply by multiplying the width and the height.

Uniform Distribution

Example 1 Suppose the time it takes a nine-year old to eat a doughnut is uniformly distributed between 0.5 and 4 minutes, inclusive.

Uniform Distribution

Example 1

1. In words, define the random variable X.
2. Find a and b.
3. Give the distribution of X.
4. Find the probability density function. Draw a diagram.
5. Find The probability that a randomly selected nine-year old child eats a doughnut in at least two minutes $P(X > 2)$. Draw a diagram.
6. Find the probability that a different nine-year old child eats a doughnut in more than two minutes given that the child has already been eating the doughnut for more than 1.5 minutes. Draw a diagram.

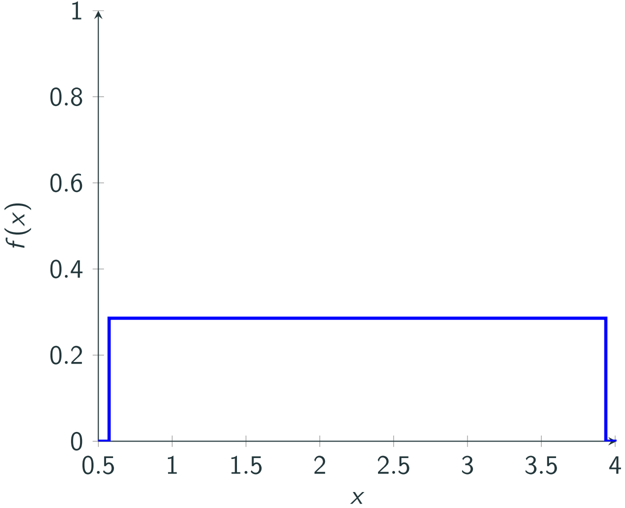
Uniform Distribution

Example 1 - Solution

1. X is the amount of time it takes for a nine year old to eat a doughnut.
2. $a = 0.5$, $b = 4$
3. $X \sim Uniform(0.5, 4)$
4. $f_x(x) = \dfrac{1}{4 - 0.5}$
5. $P(X > 2) = height * width = \dfrac{1}{(4 - 0.5)}(4 - 2) = 0.5714$

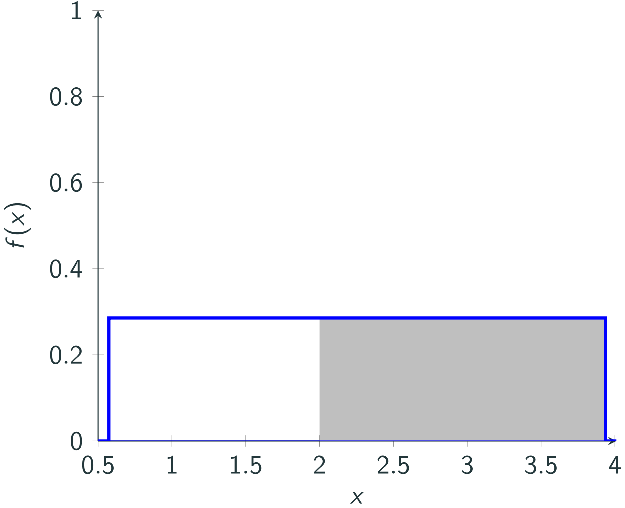
Uniform Distribution

Example 1 - Solution (4)



Uniform Distribution

Example 1 - Solution (5)



Uniform Distribution

Example 1 - Solution

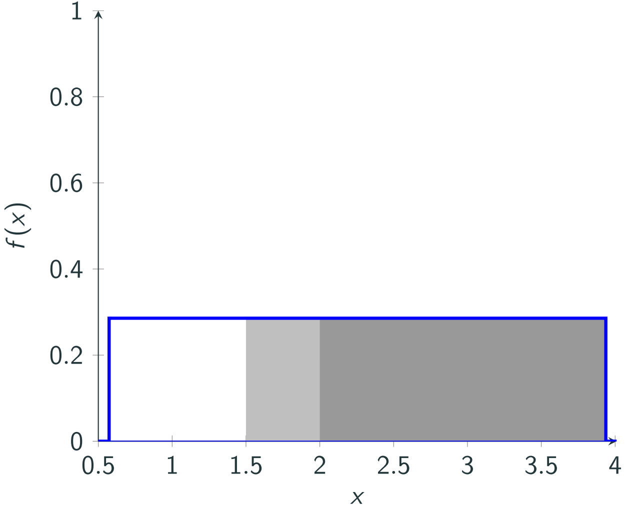
6. This is a conditional probability problem. We need to find $P(X > 2|X > 1.5)$

   General formula for Conditional Probability: $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$.
   Consider $P(X > 1.5)$ as Event A and $P(X > 2)$ as Event B.

$$P(X > 2|X > 1.5) = \frac{\frac{1}{4-0.5}(4-2)}{\frac{1}{4-0.5}(4-1.5)} = \frac{2}{2.5} = 0.8$$

Uniform Distribution

Example 1 - Solution (6)



Uniform Distribution

Example 2 The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between zero and 15 minutes, inclusive.

Uniform Distribution

Example 2

1. In words, define the random variable X.
2. Give the distribution of X
3. Find a, b, μ and σ.
4. Find the probability density function. Draw a diagram.
5. What is the probability that a person waits fewer than 12.5 minutes? Draw a diagram.
6. Ninety percent of the time, the time a person must wait falls below what value? Draw a diagram.

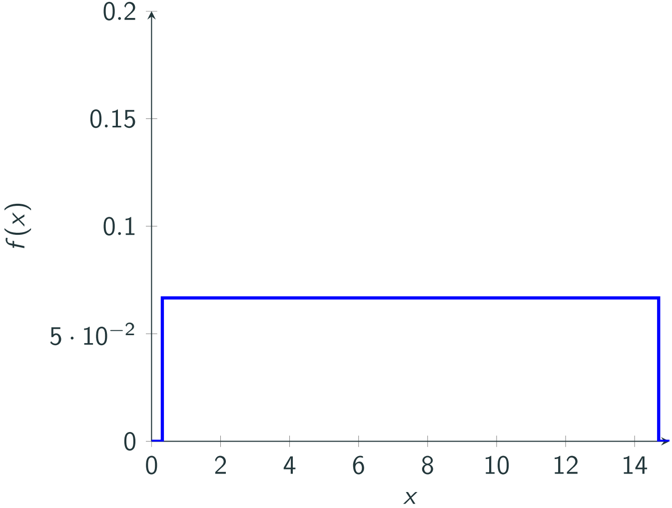
Uniform Distribution

Example 2 - Solution

1. X is the amount of time that a person must wait for the bus.
2. $X \sim Uniform(0, 15)$
3. $a = 0$, $b = 15$, $\mu = \dfrac{0 + 15}{2}$ and $\sigma = \sqrt{\dfrac{(15 - 0)^2}{12}} = 4.33$
4. $f_x(x) = \dfrac{1}{15 - 0}$
5. $P(X < 12.5) = height * width = \dfrac{1}{(15 - 0)}(12.5 - 0) = 0.8333$

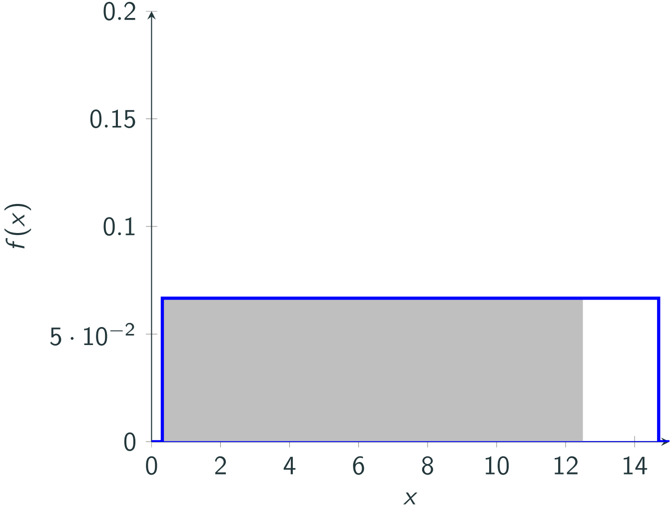
Uniform Distribution

Example 2 - Solution (4)



Uniform Distribution

Example 2 - Solution (5)



Uniform Distribution

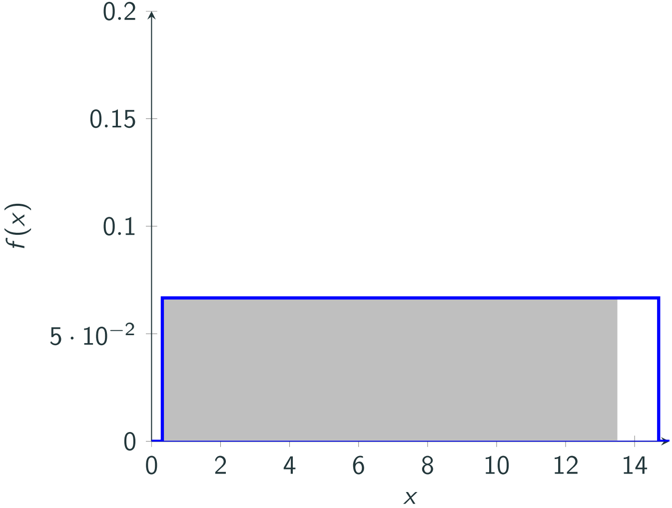
Example 2 - ***Solution***

6. Here we have find an X such that the area is 90 percent or 0.90.

$$P(X < x) = height * width = 0.90 = \frac{1}{(15-0)}(x-0) = 13.5$$

Uniform Distribution

Example 2 - Solution (6)



Exponential Distribution

Definition A statistical experiment is defined as an exponential experiment when a continuous random variable X has an exponential distribution with mean μ then the decay parameter is $\lambda = \frac{1}{\mu}$.

Exponential Distribution

- Distribution: $X \sim Exp(\lambda)$
  - where $\mu \geq 0$
  - where $\lambda \geq 0$
- Mean: $\mu = \frac{1}{\lambda}$
- Standard Deviation: $\sigma = \frac{1}{\lambda}$
- Probability Density Function: $f_x(x) = \lambda \exp^{(-\lambda x)}$
- Probability: $P(X \leq x) = 1 - \exp^{(-\lambda x)}$

Exponential Distribution

Example 3 On the average, a certain computer part lasts ten years. The length of time the computer part lasts is exponentially distributed.

Exponential Distribution

Example 3

1. In words, define the random variable X.
2. Give the distribution of X.
3. Find μ and λ.
4. Find the probability density function. Draw a diagram.
5. What is the probability that a computer part lasts more than 7 years?
6. Eighty percent of computer parts last at most how long?

Exponential Distribution

Example 3 - Solution

1. X is the amount of time that a computer part lasts.

2. $X \sim Exp(\frac{1}{10})$

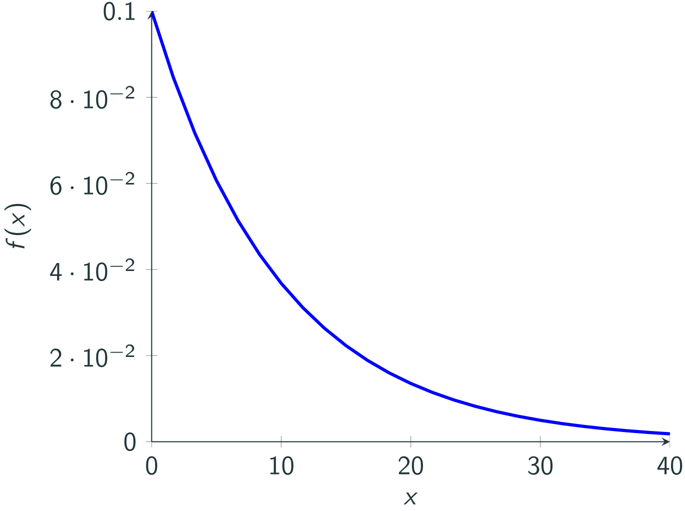
3. $\mu = 10$, $\lambda = \frac{1}{10} = 0.1$

4. $f_x(x) = \frac{1}{10} exp^{(-\frac{x}{10})}$

5. $P(X > 7) = 1 - P(X \leq 7) = 1 - (1 - \exp^{(-\frac{7}{10})}) = 0.4966$

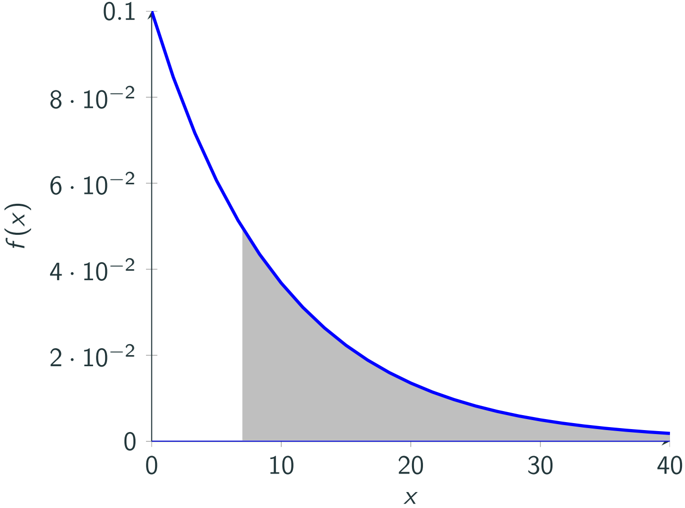
Exponential Distribution

Example 3 - Solution (4)



Exponential Distribution

Example 3 - Solution (5)



Exponential Distribution

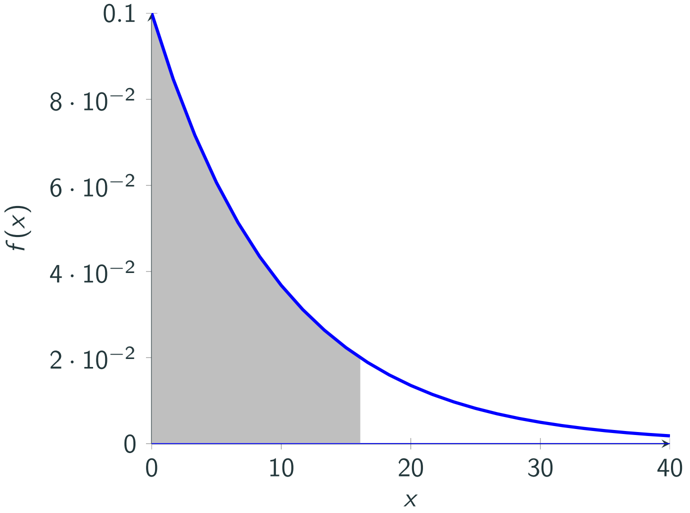
Example 3 - Solution

6. Here we have find an X such that the area is 80 percent or 0.80.

$$
\begin{aligned}
P(X < x) &= 0.80 = 1 - \exp^{(-\lambda x)} \\
&\Rightarrow \exp^{(-\lambda x)} = 0.20 \\
&\Rightarrow -\lambda x \log \exp = \log 0.20 \\
&\Rightarrow -0.1x = -1.609 \\
&\Rightarrow x = \frac{1.609}{0.1} \\
&= 16.09
\end{aligned}
$$

Exponential Distribution

Example 3 - Solution (6)



Exponential Distribution

Example 4 Suppose that the length of a phone call, in minutes, is an exponential random variable with decay parameter,$\lambda = \frac{1}{12}$.

Exponential Distribution

Example 4

1. In words, define the random variable X.
2. Give the distribution of X.
3. Find μ.
4. Find the probability density function. Draw a diagram.
5. Find the probability that you will have to wait more than five minutes. Draw a diagram.
6. Find the probability that you will have to wait between seven and thirteen minutes. Draw a diagram.

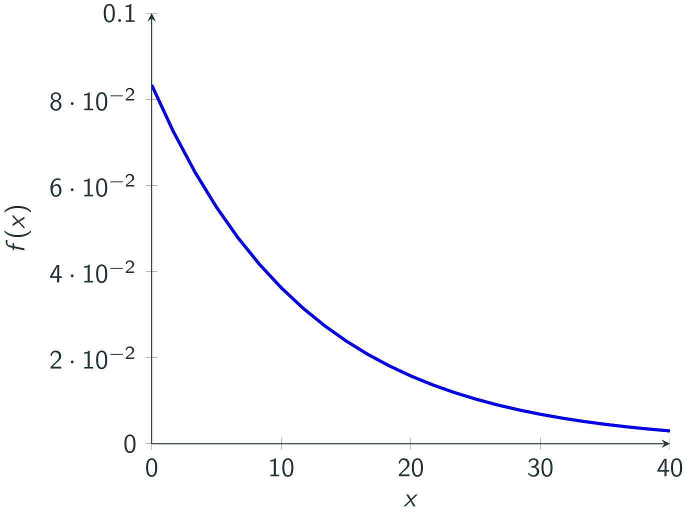
Exponential Distribution

Example 4 - Solution

1. X is the amount of time that a phone call lasts.
2. $X \sim Exp(\frac{1}{12})$
3. $\mu = \frac{1}{\lambda} = 12$
4. $f_x(x) = \frac{1}{12} exp^{(-\frac{x}{12})}$
5. $P(X > 5) = 1 - P(X \leq 5) = 1 - (1 - \exp^{(-\frac{5}{12})}) = 0.6592$

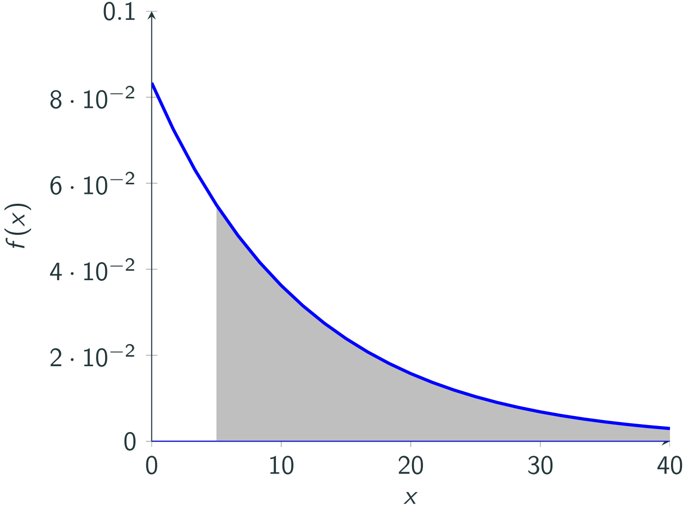
Exponential Distribution

Example 4 - Solution (4)



Exponential Distribution

Example 4 - Solution (5)



Exponential Distribution

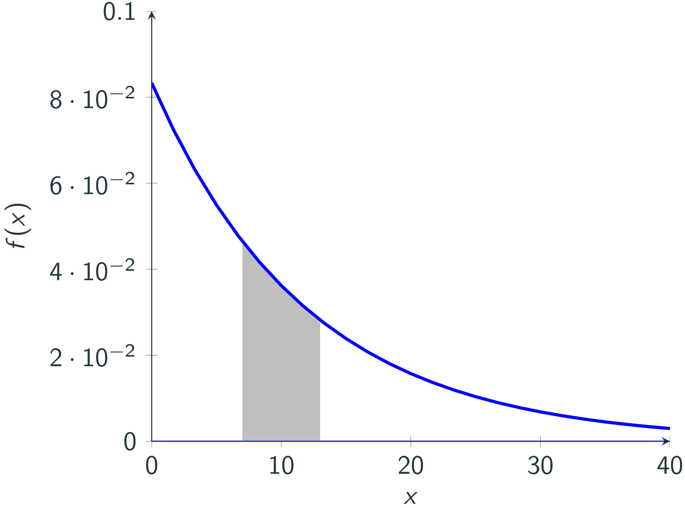
Example 4 - Solution

6. First we will find $P(X < 13)$ and then we will find $P(X < 7)$. Finally we calculate $P(7 < X < 13) = P(X < 13) - P(X < 7)$.

$$\begin{aligned} P(X < 13) &= 1 - \exp^{(-\frac{13}{12})} = 0.6615 \\ P(X < 7) &= 1 - \exp^{(-\frac{7}{12})} = 0.4419 \\ P(7 < X < 13) &= P(X < 13) - P(X < 7) = 0.2195 \end{aligned} \tag{1}$$

Exponential Distribution

Example 4 - Solution (6)



Next class we will cover Normal Distribution and Index Numbers.