Expectation and Variance

Discrete Random Variables

Noara Razzak

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What is a Random variable?

A random variable is **a bridge (function) between real-world randomness and numbers**. It lets us:

- Model uncertainty (e.g., "Will it rain?" $\rightarrow X = 1$ if yes, 0 if no).
- Mathematically explain daily randomness (e.g., "What's the average wait time?").

Real-World Analogies

Random Variable	**Real-World Meaning**
X: "Roll of a die"	Outcome of a game move
Y: "Daily stock price change"	Profit/loss from investing
Z: "Height of a random person"	Variability in a population

Common Misconceptions of Random Variables

- **"It's just a variable"**: No, it's a *function* that assigns numbers to outcomes.
- **"It's always unknown"**: We often know its *possible values* and *probabilities* (e.g., a fair die).
- **"It's the same as probability"**: Probability describes random variables; it's not the variable itself.

Expectation of a random variable

- The mean, expected value, or expectation of a random variable X is written as $E(X)$ or μ_x.
- If we observe N random values of X, then the mean of the N values will be approximately equal to $E(X)$ for large N.
- The expectation is defined differently for continuous and discrete random variables

Expectation of X

Definition: Let X be a discrete random variable with probability function $f_x(x)$. The expected value of X is:

$$E(X) = \sum_x x.f_x(x) = \sum_x x.P(X = x).$$

- Here $f_x(x)$ is the probability mass function.
- It is a function that gives the probability that a discrete random variable is exactly equal to some value.

Expectation of g(X)

Definition: Let X be a discrete random variable, and let g be a function. The expected value of $g(X)$ is:

$$E(g(X)) = \sum_{x} g(X).f_x(x) = \sum_{x} g(X).P(X = x).$$

Expectation

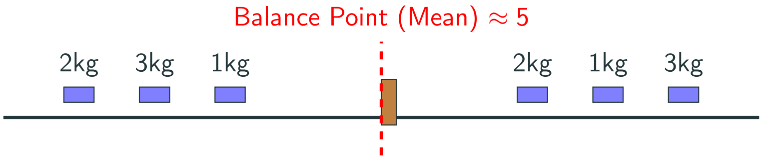
So is there an intuitive way to understand Expectation of a Random Variable?

- Expectation is the average value over many repetitions.
- It's the balance point of the distribution.
- It gives the fair price in a betting scenario.
- It's the best constant predictor under squared loss.

Careful! It doesn't always represent the most likely outcome!

Expectation For a data set $x_1, x_2, \ldots, x_n$, the mean is:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \cdots + x_n}{n}$$



Expectation

Example 1

Suppose we have a six-sided die marked with five 3's and one 6. What would you expect the average of 6000 rolls to be? What will be the variance?

Expectation

Example 1- Solution

- Notice that throwing a die is a completely random process, so we are dealing with a random variable.
- Since there are five 3's and one six we expect roughly 5/6 of the rolls will give 3 and 1/6 will give 6. Assuming this to be exactly true, we have the following table of values and counts:

value	3	6
count	$\frac{5000}{6000}$	$\frac{1000}{6000}$

Expectation

Example 1- Solution

Therefore there will be 5000/6000 probability that the die will give 3 and 1000/6000 probability that the die will give 6.

$$E(X) = \sum_{x} x.f_x(x) = \frac{5000}{6000}.3 + \frac{1000}{6000}.6 = 3.5$$

Expectation

Example 2

We roll two standard 6-sided dice. You win Taka 1000 if the sum is 2 and lose Taka 100 otherwise. How much do you expect to win on average per trial? What will be the variance?

Expectation

Example 2-Solution

- Notice that you are tossing two die. Therefore, there are 36 total outcomes.
- Sum of the outcomes can be numbers between 2 and 36.
- There is only one outcome where the number is 2. Each dice can give 1 only once.

Expectation

Example 2 -Solution

Suppose you try tossing the two die N number of times. The outcome will be 2 about $\frac{N}{36}$ of the times and other than 2 about $\frac{35.N}{36}$ of the times.

value	1000	100
count	$\frac{N}{36}$	$\frac{35.N}{36}$

Therefore Expected Average:

$$E(X) = \sum_{x} x.f_x(x) = \left(\frac{N}{36}.1000 - \frac{35.N}{36}.100\right).\frac{1}{N} = -69.44$$

Expectation

Example 3

Flip a fair coin two times. Let X be the number of heads.Find the average number of heads in the coin toss. What will be the variance?

Expectation

Example 3 - Solution

- Notice that we have four outcomes- {HH, HT,TH, TT}
- Total number of outcomes is 4.

value	2H	1H	0H
count	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Expectation

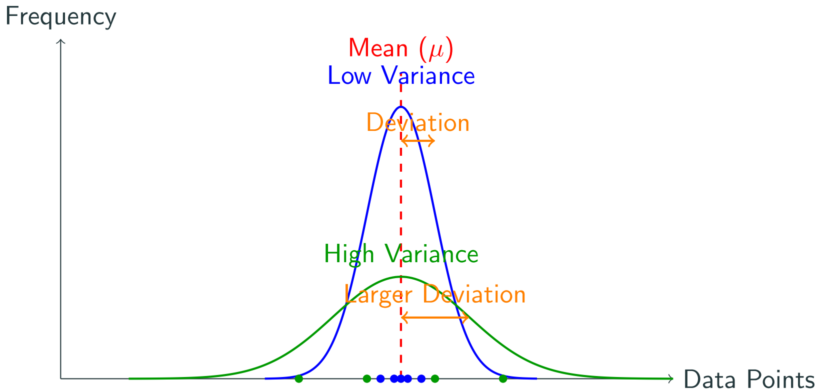
Example 3- Solution

$$E(X) = \sum_x x.f_x(x) = 2.\frac{1}{4} + 1.\frac{2}{4} + 0.\frac{1}{4} = 1$$

Variance

- The variance is the mean squared deviation of a random variable from its own mean.
- If X has high variance, we can observe values of X a long way from the mean.
- If X has low variance, the values of X tend to be clustered tightly around the mean value.

Variance (σ^2) measures how far a set of numbers are spread out from their mean value.



Variance

Definition: Let X be any discrete random variable with mean μ. The variance of X is:

$$\begin{aligned} Var(X) &= \sum_x (x-\mu)^2 f_x(x) \\ &= E(X-\mu)^2 \\ &= E\left[X^2 - 2\cdot X\cdot\mu + (\mu)^2\right] \\ &= E(X^2) - 2\cdot E(X)\cdot\mu + E(\mu^2) \\ &= E(X^2) - 2\cdot\mu\cdot\mu + \mu^2 \\ &= E(X^2) - E(X)^2 \end{aligned}$$

Variance

Example 1 - Solution Continued

Now we will find the variance.

$$
\begin{aligned}
Var(X) &= \sum_x (x-\mu)^2 f_x(x) \\
&= (3-3.5)^2.\frac{5000}{6000} + (6-3.5)^2.\frac{1000}{6000} \\
&= 1.25
\end{aligned}
$$

Variance

Example 2 - Solution Continued

$$
\begin{aligned}
Var(X) &= \sum_x (x-\mu)^2 f_x(x) \\
&= (1000+69.44)^2 . \frac{1}{36} - (100+69.44)^2 . \frac{35}{36} \\
&= 2239.30
\end{aligned}
$$

Variance

Example 3 - Solution Continued

$$
\begin{aligned}
Var(X) &= \sum_x (x-\mu)^2 f_x(x) \\
&= (2-1)^2.\frac{1}{4} + (1-1)^2.\frac{1}{2} + (0-1)^2.\frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

Expected Value

- Synonyms: Average or Mean
- Notation: $E(X)$, μ
- Definition: $\sum_x x.f_x(x)$
- Scale and Shift: $E(aX + b) = aE(X) + b$
- Additive Property: $E(X + Y) = E(X) + E(Y)$ for any r.v X and Y or their functions.
- Multiplicative Property: $E(XY) = E(X) \cdot E(Y)$ for independent any r.v X and Y or their functions.

Variance

- Synonyms: Distance from Mean
- Notation: $Var(X)$
- Definition: $\sum_x (x - \mu)^2 . f_x(x)$
- Scale and Shift: $Var(aX + b) = a^2 . Var(X)$
- Additive Property 1: $Var(X + Y) = Var(X) + Var(Y)$ if X and Y are independent r.v.s
- Additive Property 2: $Var(X + Y) = Var(X) + Var(Y) - 2Cov(X, Y)$ if X and Y are not independent r.v.s where $Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$.

Next class we will discuss expectation and variance of random variables for continuous distributions.