Expectation and Variance

Continuous Random Variables

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Expectation of X

Definition: Let X be a continuous random variable with probability function $f_x(x)$. The expected value of X is:

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f_x(x)\,\mathrm{dx}$$

- Here $f_x(x)$ is the probability density function.
- It is a function of a continuous random variable, whose integral across an interval gives the probability that the value of the variable lies within the same interval.

Expectation of g(X)

Definition: Let X be a continuous random variable, and let g be a function. The expected value of $g(X)$ is:

$$E(g(X)) = \int_{-\infty}^{+\infty} g(X) \cdot f_x(x) \, \mathrm{dx}$$

Expectation

How is the expectation of a discrete random variable different from that of a continuous random variable?

- Essentially, it is the same, however, instead of the $\sum$ symbol we use the $\int$ symbol, which also means addition.
- We are simply summing over different probabilities in both cases.

Expectation

Example 1

Let X be a continuous random variable with p.d.f

$$f(x) = \begin{cases} x^3 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Find $E(X)$ and $Var(X)$.

Expectation

Example 1- Solution

$$\begin{aligned} E(X) &= \int_0^1 x.f_x(x)\,\mathrm{dx} \\ &= \int_0^1 x.x^3\,\mathrm{dx} \\ &= \int_0^1 x^4\,\mathrm{dx} \\ &= \frac{1}{5}x^5\Big|_0^1 \\ &= \frac{1}{5} \end{aligned}$$

Expectation

Example 2

Let $X \sim Uniform(-1, 3)$. Find $E(X)$ and $Var(X)$.

Expectation

Example 2 - Solution

First we know that the p.d.f of a Uniform Distribution $X \sim Uniform(a, b)$ is:

$$f(x) = \frac{1}{b - a}$$

Therefore,

$$f(x) = \frac{1}{3 - (-1)} = \frac{1}{4}$$

Expectation

Example 2 - Solution

$$
\begin{aligned}
E(X) &= \int_{-1}^{3} x.f_x(x)\,\mathrm{dx} \\
&= \int_{-1}^{3} x.\frac{1}{4}\,\mathrm{dx} \\
&= \frac{1}{4}\int_{-1}^{3} x\,\mathrm{dx} \\
&= \frac{1}{8}x^2\Big|_{-1}^{3} \\
&= 1
\end{aligned}
$$

Variance

Definition: Let X be a continuous random variable with mean μ. The variance of X denoted by σ^2 or $V(X)$ is defined by:

$$\sigma^2 = E(X - \mu)^2 = \int (x - \mu)^2 \, \mathrm{dx}$$

Variance

We will calculate the variance using the same method we used previously.

$$\begin{aligned}
Var(X) &= E(X-\mu)^2 \\
&= E\left[X^2 - 2\cdot X\cdot \mu + (\mu)^2)\right] \\
&= E(X^2) - 2\cdot E(X)\cdot \mu + E(\mu^2) \\
&= E(X^2) - 2\cdot \mu\cdot \mu + \mu^2 \\
&= E(X^2) - E(X)^2
\end{aligned}$$

First we will find $E(X^2)$ and then we will subtract μ^2 or $E(X)^2$ from the former value.

Variance

Example 1 - Solution Continued

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^2.f_x(x)\ \mathrm{dx} \\
&= \int_0^1 x^2.x^3\ \mathrm{dx} \\
&= \int_0^1 x^5\ \mathrm{dx} \\
&= \frac{1}{6}x^6\Big|_0^1 \\
&= \frac{1}{6}
\end{aligned}
$$

Variance

Example 1 - Solution Continued

$$E(X^2) - E(X)^2 = \frac{1}{6} - \left(\frac{1}{5}\right)^2$$
$$= 0.1267$$

Variance

Example 2 - Solution Continued

$$
\begin{aligned}
E(X^2) &= \int_{-1}^{3} x^2.f_x(x)\,\mathrm{dx} \\
&= \int_{-1}^{3} x^2.\frac{1}{4}\,\mathrm{dx} \\
&= \frac{1}{4}\int_{-1}^{3} x^2\,\mathrm{dx} \\
&= \frac{1}{12}x^3\Big|_{-1}^{3} \\
&= \frac{27}{12} - \frac{(-1)}{12} = \frac{28}{12}
\end{aligned}
$$

Variance

Example 2 - Solution Continued

$$\begin{aligned} E(X^2) - E(X)^2 &= \frac{28}{12} - (1)^2 \\ &= \frac{16}{12} \end{aligned}$$

End

Next class we will look at expectations and variances of some common distributions.