Expectation and Variance

Analysis of Common Distributions

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Expectation of Bernoulli Trials

Probability Mass Function:

$f_x(x) = p^x(1-p)^{1-x}$ for $x \,\epsilon\, \{0,1\}$

Expectation of X:

$$\begin{aligned} E(X) = \sum_{x=0}^{x=1} x\, f_x(x) &= \sum_{x=0}^{x=1} x\,[p^x(1-p)^{1-x}] \\ &= 0.\, p^0(1-p)^1 + 1.\, p^1(1-p)^0 \\ &= 0 + p \\ &= p \end{aligned}$$

Variance of Bernoulli Trials

Expectation of X^2:

$$
\begin{aligned}
E(X^2) = \sum_{x=0}^{x=1} x^2 \, f_x(x) &= \sum_{x=0}^{x=1} x^2 \left[p^x (1-p)^{1-x}\right] \\
&= 0.\, p^0 (1-p)^1 + 1.\, p^1 (1-p)^0 \\
&= 0 + p \\
&= p
\end{aligned}
$$

Variance of Bernoulli Trials

$$
\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= p - p^2 \\
&= p(1-p)
\end{aligned}
$$

Expectation of Binomial Distribution

Probability Mass Function:

$f_x(x) = \binom{n}{x} \cdot p^x(1-p)^{n-x}$ for $x \,\epsilon\, W$ where W are all the natural numbers.

Expectation of X:

$$E(X) = \sum_{x=0}^{x=n} x\, f_x(x)$$
$$= \sum_{x=0}^{x=n} x \binom{n}{x} \cdot p^x(1-p)^{n-x}$$

Expectation of X:

But this is too difficult. Therefore recall that the Binomial Distribution is the sum of Bernoulli trials. Therefore, $X = \sum_{i=1}^{n} Y_i$ where X is the Binomial random variable and Y_i is the Bernoulli random variable.

$$\begin{aligned} E(X) &= E\left[\sum_{i=1}^{n} Y_i\right] \\ &= \sum_{i=1}^{n} E[Y_i] \\ &= \sum_{i=1}^{n} p \\ &= np \end{aligned}$$

Variance of Binomial Distribution

We will calculate the variance of a Binomial Distribution in the same way that we calculated the expectation.

$$\begin{aligned} Var(X) &= Var\left[\sum_{i=1}^{n} Y_i\right] \\ &= \sum_{i=1}^{n} Var[Y_i] \\ &= \sum_{i=1}^{n} p(1-p) \\ &= np(1-p) \end{aligned}$$

Expectation of Geometric Distribution

Probability Mass Function:

$f_x(x) = p(1-p)^x$ for $x \epsilon N$ where N are all the natural numbers.

Expectation of X:

$$\begin{aligned} E(X) &= \sum_{x=0}^{\infty} x \cdot f_x(x) \\ &= \sum_{x=0}^{\infty} x \cdot p(1-p)^{x-1} \\ &= p \cdot \sum_{x=0}^{\infty} x \cdot (1-p)^{x-1} \end{aligned}$$

Expectation of Geometric Distribution

Notice that $\sum_{n=0}^{\infty} y^n = \dfrac{1}{1-y}$.

This is the Taylor Series Expansion of $y \epsilon (0, 1)$. Differentiating both sides with $\dfrac{d}{dy}$:

$$\sum_{n=0}^{\infty} n \cdot y^{n-1} = \frac{1}{(1-y)^2}$$

. Now put $n = x$ and $y = (1-p)$, so we get:

$$\sum_{x=0}^{\infty} x \cdot (1-p)^{x-1} = \frac{1}{(1-(1-p))^2} = \frac{1}{p^2}$$

Expectation of Geometric Distribution

Expectation of X***:*** We put the value found in the previous slide on this slide.

$$
\begin{aligned}
E(X) &= p \cdot \sum_{x=0}^{\infty} x \cdot (1-p)^{x-1} \\
&= p \cdot \frac{1}{p^2} \\
&= \frac{1}{p}
\end{aligned}
$$

Expectation of Uniform Distribution

Probability Density Function:

$f_x(x) = \dfrac{1}{b-a}$ for $a, b \in R$ where R are all the real numbers.

Expectation of X:

$$\begin{aligned} E(X) &= \int x \cdot f_x(x)\,\mathrm{dx} \\ &= \int_a^b x \cdot \frac{1}{b-a}\,\mathrm{dx} \\ &= \frac{1}{(b-a)} \int_a^b x\,\mathrm{dx} \end{aligned}$$

Expectation of X:

$$E(X) = \frac{1}{(b-a)}\left[\frac{1}{2}\cdot x^2\Big|_a^b\right]$$

$$= \frac{1}{(b-a)}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{(b-a)}\cdot\frac{(b+a)(b-a)}{2}$$

$$= \frac{(b+a)}{2}$$

Expectation of Exponential Distribution

Probability Density Function:

$f_x(x) = \lambda \cdot \exp^{-\lambda x}$ for $x \, \epsilon \, R_+$ where R_+ are all the positive real numbers.

Expectation of X:

$$
\begin{aligned}
E(X) &= \int x \cdot f_x(x) \, \mathrm{dx} \\
&= \int_0^\infty x \cdot \lambda \cdot \exp^{-\lambda x} \, \mathrm{dx}
\end{aligned}
$$

Expectation of Exponential Distribution

We will use by parts integration to solve this problem.

$$\int v \, \mathrm{du} = uv - \int u \, \mathrm{dv}$$

$$\begin{aligned} &v = x \\ &dv = dx \\ &u = \exp^{-\lambda x} \\ &du = -\lambda \cdot \exp^{-\lambda x} \; dx \end{aligned} \tag{1}$$

Expectation of Exponential Distribution

$$E(X) = \int_0^\infty x \cdot \lambda \cdot \exp^{-\lambda x} \ \mathrm{dx}$$

$$= -\left[x \cdot \exp^{-\lambda x} \Big|_0^\infty\right] + \int_0^\infty \exp^{-\lambda x} \ \mathrm{dx}$$

$$= -\left[\infty \cdot (\exp^{-\lambda \cdot \infty}) - 0 \cdot (\exp^{-\lambda \cdot 0})\right] + \left[\frac{-1}{\lambda} \exp^{-\lambda x} \Big|_0^\infty\right]$$

$$= \left[\left(\frac{-1}{\lambda} \exp^{-\lambda \cdot \infty}\right) - \left(\frac{-1}{\lambda} \exp^{-\lambda \cdot 0}\right)\right]$$

$$= \frac{1}{\lambda}$$

Expectation of Standard Normal Distribution

Probability Density Function:

$f_x(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp^{-\frac{1}{2}x^2}$ for $x \epsilon R$ where R are all the real numbers.

Expectation of X:

$$\begin{aligned} E(X) &= \int x \cdot f_x(x) \,\mathrm{dx} \\ &= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}x^2} \,\mathrm{dx} \end{aligned}$$

Expectation of Standard Normal Distribution

We will use substitution to solve this problem.

$$\begin{aligned} u &= x^2 \\ du &= 2x \, \mathrm{dx} \\ dx &= \frac{du}{2x} \end{aligned} \tag{2}$$

Expectation of Standard Normal Distribution

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}x^2} \; \mathrm{dx} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x \cdot \exp^{-\frac{1}{2}x^2} \; \mathrm{dx} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x \cdot \exp^{-\frac{1}{2}u} \; \frac{\mathrm{du}}{\mathrm{2x}} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \frac{1}{2} \cdot \exp^{-\frac{1}{2}u} \; \mathrm{du} \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} \cdot -2. \exp^{-\frac{1}{2}u} \Big|_{-\infty}^{\infty} \right]
\end{aligned}
$$

Expectation of Standard Normal Distribution

$$
\begin{aligned}
E(X) &= \frac{1}{\sqrt{2\pi}} \left[-\exp^{-\frac{1}{2}x^2} \Big|_{-\infty}^{\infty} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\exp^{-\frac{1}{2}\infty} - \left(-\exp^{-\frac{1}{2}\infty} \right) \right] \\
&= \left[-\exp^{-\frac{1}{2}\infty} + \exp^{-\frac{1}{2}\infty} \right] \\
&= 0
\end{aligned}
$$

Expectation of Normal Distribution

The Normal random variable X is expressed as $X = \mu + \sigma Z$ where Z is the Standard Normal Distribution.

$$\begin{aligned} E(X) &= E(\mu + \sigma Z) \\ &= \mu + \sigma \cdot E(Z) \\ &= \mu + \sigma \cdot 0 \\ &= \mu \end{aligned}$$

End

We will not discuss the variance of every distribution because the calculations are too cumbersome, however you can practice on your own calculation using $Var(X) = E(X^2) - E(X)^2$ ***for each case.***