Hypothesis Testing

With two samples

Noara Razzak

May 9, 2020

Basic Concepts - Review from last lecture

- Null Hypothesis (H_0): Default assumption (e.g., $\mu = \mu_0$)
- Alternative Hypothesis (H_1): Counter-claim (e.g., $\mu \neq \mu_0$)
- Significance Level (α): Probability of Type I error (typically 0.05 but we can also use 0.01)

Types of Tests

Z-test (Known Variance)

$$T_n = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

Student's t-test (Unknown Variance)

$$T_n = \frac{\bar{X}_n - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

Decision Rules

Test Type	Rejection Region	Condition
Two-tailed	$\lvert T_n \rvert > z_{\alpha/2}$	$H_1 : \mu \neq \mu_0$
Right-tailed	$T_n > z_\alpha$	$H_1 : \mu > \mu_0$
Left-tailed	$T_n < -z_\alpha$	$H_1 : \mu < \mu_0$

Error Types

$$\begin{cases} \text{Type I Error} = P(\text{Reject } H_0 \mid H_0 \text{ true}) = \alpha \\ \text{Type II Error} = P(\text{Fail to reject } H_0 \mid H_1 \text{ true}) = \beta \end{cases}$$

Two Population Means with Unknown Standard Deviations

When comparing two independent population means where the population standard deviations are unknown, we use the two-sample (Welch) t-test.

When to use:

- Used when there are two populations with mean μ_1 and μ_2 and variances are unknown
- Welch's t-test applied with different degrees of freedom
- The test examines whether $\mu_1 = \mu_2$ (null hypothesis)
- Sample sizes n_1 and n_2 respectively
- Sample means $\bar{X}_1$ and $\bar{X}_2$
- Sample standard deviations s_1 and s_2
- Assumptions: Normally distributed populations or large sample sizes (CLT)

Welch's Test statistic

The test statistic is:

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{1}$$

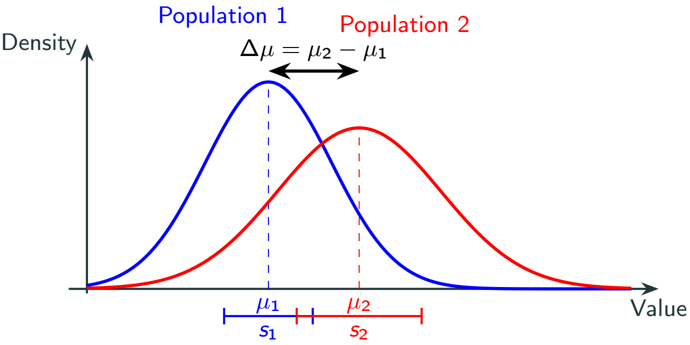
Where s_p is the pooled standard deviation:

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \tag{2}$$

Under the null hypothesis ($H_0 : \mu_1 = \mu_2$).
Degrees of freedom:

$$df = n_1 + n_2 - 2 \tag{3}$$

Visual Representation



Example 1

A study is done by a community group in two neighboring colleges to determine which one graduates students with more math classes. College A samples 11 graduates. Their average is four math classes with a standard deviation of 1.5 math classes. College B samples nine graduates. Their average is 3.5 math classes with a standard deviation of one math class. The community group believes that a student who graduates from college A has taken more math classes, on the average. Both populations have a normal distribution. Test at a 1% significance level. Answer the following questions.

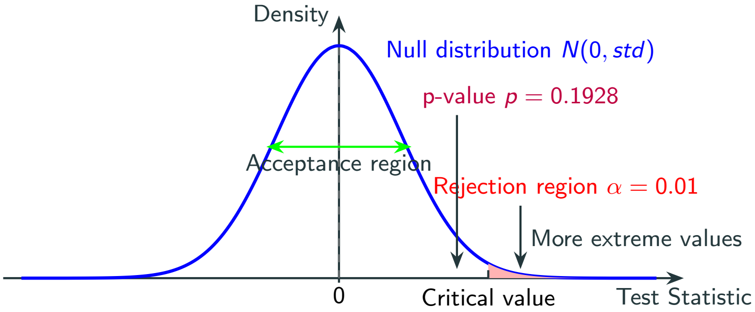
Example 1

- Are the populations standard deviations known or unknown?
- Which distribution do you use to perform the test
- What is the random variable?
- What are the null and alternate hypotheses? Write the null and alternate hypotheses in words and in symbols.
- Is this test right-, left-, or two-tailed?
- What is the p-value?
- Do you reject or not reject the null hypothesis?

Solution

- Two means
- Unknown
- Student's t- test
- $\bar{X}_A - \bar{X}_B$, it's the difference between the number of math classes from Colleges A and B respectively
- $H_o : \mu_A \geq \mu_B$ and $H_A : \mu_A < \mu_B$
- It's a right-tailed test with $\alpha = 0.01$
- The p value is 0.1928
- Do not reject null hypothesis, since $\alpha = 0.01 < 0.1928$.

One-tailed test, with $\alpha = 0.01$



Two Population Means with Known Standard Deviations

When comparing two independent population means where the population standard deviations are known, we use the two-sample z-test.

When to use:

Given:

- Used when there are two populations with mean μ_1 and μ_2 and variances are known
- The normal distribution is applied rather than the t-distribution
- The test examines whether $\mu_1 = \mu_2$ (null hypothesis)
- Sample sizes n_1 and n_2 respectively
- Sample means $\bar{X}_1$ and $\bar{X}_2$
- Population standard deviations known, σ_1 and σ_2
- Test statistic follows standard normal distribution N(0,1) under H_0

Test statistic

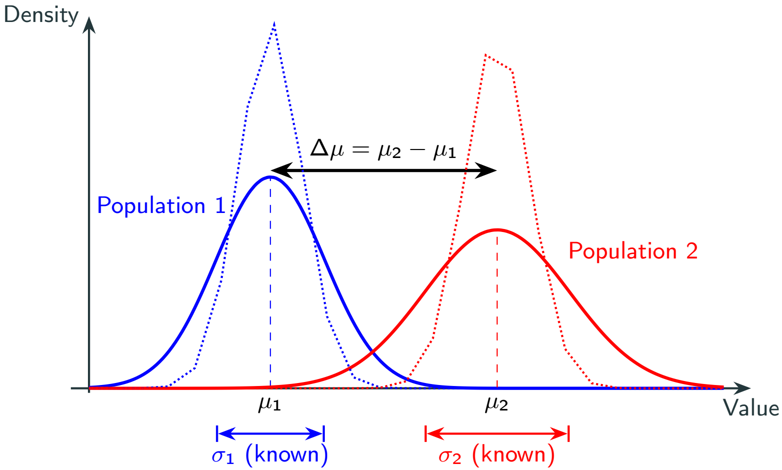
The test statistic is

$$z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \tag{4}$$

Under the null hypothesis ($H_0 : \mu_1 = \mu_2$):

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \tag{5}$$

Visual Representation



Example 2

The mean lasting time of two competing floor waxes is to be compared. Twenty floors are randomly assigned to test each wax. Both populations have a normal distributions. The data are recorded in the following table:

Table 1: Comparison of Floor Wax Effectiveness

Component	**Wax 1**	**Wax 2**
Sample Mean ($\bar{X}$)	3.0 months	2.9 months
Population SD (σ)	0.33	0.36

Does the data indicate that wax 1 is more effective than wax 2? Test at a 5% level of significance.

Example 2

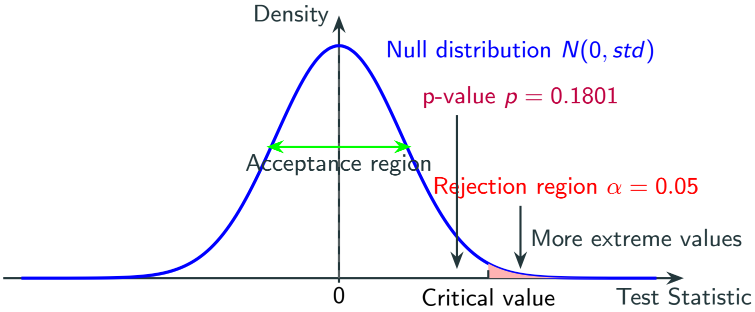
- Are the populations standard deviations known or unknown?
- Which distribution do you use to perform the test?
- What is the random variable?
- What are the null and alternate hypotheses? Write the null and alternate hypotheses in words and in symbols.
- Is this test right-, left-, or two-tailed?
- What is the z- score and corresponding p-value?
- Do you reject or not reject the null hypothesis?

Solution

- Two means
- Known
- Normal Distribution, $\bar{X_1} - \bar{X_2} \sim N\left(0, \sqrt{\frac{0.33^2}{20} + \frac{.036^2}{20}}\right)$
- $\bar{X_1} - \bar{X_2}$, it's the difference in the mean number of months the competing floor waxes last.
- $H_o : \mu_1 \leq \mu_2$ and $H_A : \mu_1 > \mu_2$
- It's a right-tailed test with $\alpha = 0.05$

- We need to text whether$X_1 - X_2 \leq 0$
- The test statistic is: $\frac{X_1 - X_2}{\sigma} = \frac{0.1}{0.1092} = 0.915$
- The corresponding p-value is 0.1801
- Do not reject null hypothesis, because $\alpha = 0.05 < 0.1801$.

One-tailed test, with $\alpha = 0.05$



Next class we will cover The Chi-Square Distribution, Goodness-of-Fit Test, Test of Independence, Test for Homogeneity